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EQUILIBRIUM PROBLEM OF A PLATE WITH AN OBLIQUE CUT

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The nonpenetration condition for a plate with an oblique cut is proposed. The variational formulation of the equilibrium problem and the equivalent formulation in the form of a boundary-value problem are obtained. The analytical solution is given for a one-dimensional case (a beam with a cut), and the qualitative properties of this solution are studied.

Introduction. The presence of a cut in a plate means that in addition to the outside edges, the plate has inside ones, which are called cut faces. In an undeformed state, the cut faces are in contact with each other everywhere along the two-dimensional surface, determining the shape of the cut. If, for the external faces, one can impose, for example, a jam condition, then for the cut faces, it is natural to assume the possibility of contact along the cut surface and to require their mutual nonpenetration. The restrictions that characterize this class of displacements of the points of the cut faces will be called a nonpenetration condition. This condition can incorporate the friction between the cut faces during their contact as well.

One can use the term crack instead of the term cut, assuming that the crack has a zero opening in an undeformed state. However, attention in the existing theory of cracks is mainly focused on the problems of crack propagation and the determinations of the quantities that characterize the deformed state [1]. In this case, the boundary conditions considered at the crack sides usually imply violation of the nonpenetration condition [2].

We consider the problem of finding the displacement field of the cut-containing plate's points with allowance for the nonpenetration condition, which leads to the variational and boundary-value formulations. As the analytical and numerical results obtained by Kovtunenko [3, 4] show, taking into account the nonpenetration condition changes significantly the qualitative character of the solution for thin plates (Kirchhoff model).

The problems with cuts have wide applications not only in designing structures, but, for example, in geology: cuts can simulate faults of tectonic plateforms described by thin plates in tectonics [5]. The nonpenetration condition for thin plates with cuts and the variational formulation of the equilibrium problem with a cut were proposed and studied by Khludnev and Sokolowski for the first time [6]. Khludnev [7] considered a cracked shell and studied control in the problem in which a crack opening serves as the optimality criterion. The contact problem for a cracked plate with a rigid punch was studied in [8]. An analytical solution for the one-dimensional case (for the problem of a beam with a cut) was constructed in [3]. An algorithm for numerical solution of the problem of a plate with a cut was proposed in [4].

In the present paper, a nonpenetration condition for a plate with an oblique cut that generalizes the case of a vertical cut is proposed. Variational and equivalent differential formulations of the problem are given. The problem is solved analytically for the one-dimensional case (a beam with a cut), and qualitative properties of the solution are examined.

1. Nonpenetration Condition. We denote the horizontal and vertical displacements of the points $x = (x_1, x_2)$ of the plate's median plane Ω by $W(x) = (u_1(x), u_2(x))$ and $w(x)$, respectively. Here $\Omega \subset R^2$ is the bounded domain with a smooth boundary. Let $2h$ be the plate thickness. According to the hypothesis of

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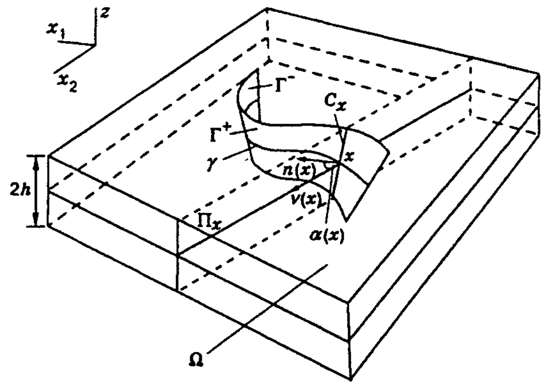

Fig. 1

straight normals, adopted in the theory of thin plates, the vertical displacements $w(x, z)$ of the positions of the plate which are at a distance z from the middle plane are considered equal to $w(x)$; horizontal displacements in a z-linear approximation are found from the relations [9]

$$W(x,z) = W(x) + z\nabla w(x), \qquad |z| \leqslant h.$$

Let the plate have a zero-width cut which is through over the thickness and does not reach the lateral boundaries and which is described by a sufficiently smooth surface Γ having no self-crossings (Fig. 1). Introducing the vector of unit normal $n(x, z)$ at each point $(x, z) \in \Gamma$, we define the positive Γ^+ and negative Γ^- cut faces. We denote the angle between $n(x, z)$ and the middle plane Ω by $\alpha(x, z)$, the curve obtained by the intersection of Γ and Ω by γ, and the middle plane of the plate with a cut by Ω_γ. Next, we define the unit vector $(\nu_1(x), \nu_2(x), 0)$, whose direction coincides with the projection of $n(x, 0)$ onto Ω_γ, and introduce the notation $\nu = (\nu_1, \nu_2)$. We then have $n(x, 0) = (\nu(x)\cos\alpha(x, 0), \sin\alpha(x, 0))$. Let Π_x be the vertical plane passing through the point $x \in \gamma$ in the direction of $n(x, 0)$ and C_x be the section obtained in intersecting Π_x and Γ. We assume that the segment C_x is rectilinear in each cross section of Π_x. We then have $n(x, z) = n(x, 0) \equiv n(x)$ and $\alpha(x, z) = \alpha(x, 0) \equiv \alpha(x)$ $\forall z\,(|z| \leqslant h)$, and the coordinates $(\tilde{x}, z)$ of the points of the surface Γ are found from the relations $\tilde{x} = x - z\nu(x)\tan\alpha(x)$ and $|z| \leqslant h$, $x \in \gamma$. Displacements of the middle plane's points at the positive (negative) face are denoted by $W^+(x)$ and $w^+(x)$ [respectively, $W^-(x)$ and $w^-(x)$].

In deriving displacement conditions for points Γ^+ and Γ^-, we assume the angle α to be sufficiently small. After this, we can assume, in the z-linear approximation, that all the points of the segment C_x have the same horizontal and vertical displacements, depending on the cut faces, and we set

$$w^\pm(\tilde{x}, z) = w^\pm(x), \quad x \in \gamma, \qquad W^\pm(\tilde{x}, z) = W^\pm(x) + z\nabla w^\pm(x), \quad |z| \leqslant h, \quad x \in \gamma.$$

The nonpenetration condition for the cut faces Γ^+ and Γ^- consists of the fact that the difference between their displacements at each point $(\tilde{x}, z)$ in the projection onto the normal $n(x)$ should be nonnegative. Denoting $[W(x)] = W^+(x) - W^-(x)$ and $[w(x)] = w^+(x) - w^-(x)$, $x \in \gamma$, we write this condition as a scalar product:

$$([W] + z[\nabla w], [w]) \cdot n \geqslant 0 \qquad \forall z\,(|z| \leqslant h) \text{ at each point } x \in \gamma,$$

or

$$\Phi_z\Big(W, w, \frac{\partial w}{\partial \nu}\Big) \equiv \Big[W, w, \frac{\partial w}{\partial \nu}\Big] \cdot (n, z\cos\alpha) \geqslant 0 \qquad (|z| \leqslant h) \text{ at each point } x \in \gamma. \tag{1.1}$$

Here $[W, w, \partial w/\partial \nu] \equiv ([W], [w], [\partial w/\partial \nu])$ and $\partial w/\partial \nu$ denotes the normal derivative on γ. We note that if the inequality (1.1) holds for $z = \pm h$, it is satisfied for all $z, |z| \leqslant h$ owing to its linear character in z. Thus, condition (1.1) can be written in the equivalent and z-free form:

$$\Phi\Big(W, w, \frac{\partial w}{\partial \nu}\Big) \equiv [W]\cdot \nu + [w]\tan\alpha - h\Big|\Big[\frac{\partial w}{\partial \nu}\Big]\Big| \geqslant 0 \quad \text{at each point } x \in \gamma. \tag{1.2}$$

For $\alpha(x) = 0$, we obtain, from (1.2), the known nonpenetration condition for a strictly vertical cut [6]:

$$[W]\cdot \nu \geqslant h\Big|\Big[\frac{\partial w}{\partial \nu}\Big]\Big| \quad \text{at each point } x \in \gamma.$$

Thus, condition (1.1) [or (1.2)] describes, in the z-linear approximation, the interaction of the faces of an oblique cut, admitting its contact (the equality case) or the absence of contact (the inequality case) at some previously unknown segments, and simultaneously excludes the mutual penetration of the cut faces.

2. Variational Formulation of the Problem. Under the smallness conditions for deformations, the energy functional of an isotropic plate whose middle plane occupies the domain Ω_γ is of the form [9]

$$J(W,w) = \frac{1}{2}\mathcal{A}(W,W) + \frac{1}{2}\mathcal{B}(w,w) - \langle (F,f),(W,w)\rangle_\Omega,$$

where $\langle\cdot,\cdot\rangle_\Omega$ refers to integration over the domain Ω_γ;

$$\mathcal{A}(W,\bar W) = G\int\limits_{\Omega_\gamma}\Big(u_{1,1}\bar u_{1,1} + u_{2,2}\bar u_{2,2} + æ(u_{1,1}\bar u_{2,2} + u_{2,2}\bar u_{1,1}) + \frac{1-æ}{2}(u_{1,2}+u_{2,1})(\bar u_{1,2}+\bar u_{2,1})\Big)\,d\Omega_\gamma;$$

$$\mathcal{B}(w,\bar w) = D\int\limits_{\Omega_\gamma}(w_{,11}\,\bar w_{,11} + w_{,22}\,\bar w_{,22} + æ w_{,11}\,\bar w_{,22} + æ w_{,22}\,\bar w_{,11} + 2(1-æ)w_{,12}\,\bar w_{,12})\,d\Omega_\gamma;$$

$D = Eh^3/3(1-æ^2)$, $G = Eh/(1-æ^2)$, E is the Young modulus, $æ$ is the Poisson ratio ($0 < æ < 0.5$), (F,f) is the vector of external forces, $F = (f_1, f_2)$, and $\bar W = (\bar u_1, \bar u_2)$. The subscript after the comma refers to differentiation with respect to the corresponding coordinate.

We specify the conditions at the external boundary of the domain Ω_γ: $W = w = \partial w/\partial\nu = 0$. Here the normal derivative $\partial w/\partial \nu$ refers to the external boundary of the domain Ω_γ. In the curve γ, we require the satisfaction of the nonpenetration condition for the cut faces: $\Phi_z(W, w, \partial w/\partial\nu) \geqslant 0\ \forall z$, $|z| \leqslant h$.

Let $H^1_{0,\gamma}(\Omega_\gamma)$ be the subspace of the Sobolev space $H^1(\Omega_\gamma)$ which consists of the functions vanishing at the external boundary of the domain Ω_γ, and $H^2_{0,\gamma}(\Omega_\gamma)$ be the subspace of the Sobolev space $H^2(\Omega_\gamma)$ which consists of the functions vanishing together with the first derivative at the external boundary of Ω_γ.

We introduce the space $H = H^1_{0,\gamma}(\Omega_\gamma) \times H^1_{0,\gamma}(\Omega_\gamma) \times H^2_{0,\gamma}(\Omega_\gamma)$ and the set $K = \{(W,w) \in H \,|\, \Phi_z(W,w,\partial w/\partial\nu) \geqslant 0\ \forall z,\ |z| \leqslant h\}$. It is then possible to consider the nonpenetration condition almost everywhere in γ. We shall assume that $F \in L^2(\Omega_\gamma)\times L^2(\Omega_\gamma)$, $f \in L^2(\Omega_\gamma)$. The equilibrium problem of a plate with the cut γ under the condition of mutual nonpenetration can be formulated as a minimization problem:

$$\inf_{(W,w)\in K} J(W,w) = J(W^*, w^*). \tag{2.1}$$

The functional J is coercive, weakly semicontinuous from below, and strictly convex on H. The set K coincides with the closed convex set $\{(W,w) \in H \,|\, \Phi(W,w,\partial w/\partial\nu) \geqslant 0\}$ owing to the equivalence of (1.1) and (1.2). Hence, the minimization problem (2.1) has a unique solution [10], which is denoted by $\{W^*, w^*\}$.

3. Formulation in the Form of a Boundary-Value Problem. We introduce the strain tensor $\varepsilon_{ij}(W) = (1/2)(u_{i,j} + u_{j,i})$ $(i,j = 1)$. According to Hook's law, the stress tensor $\sigma_{ij}(W)$ $(i,j = 1)$ is of the following form for a homogeneous isotropic plate:

$$\sigma_{11}(W) = G(\varepsilon_{11}(W) + æ\varepsilon_{22}(W)),\quad \sigma_{22}(W) = G(\varepsilon_{22}(W) + æ\varepsilon_{11}(W)),\quad \sigma_{12}(W) = G(1-æ)\varepsilon_{12}(W).$$

We determine, in γ^+, the stress $\sigma^+(W) = (\sigma_{1j}(W^+)\nu_j, \sigma_{2j}(W^+)\nu_j)$, the transverse force

$$t^+(w) = D\frac{\partial}{\partial\nu}\Big(\Delta w^+ + (1-æ)\frac{\partial^2 w^+}{\partial\tau^2}\Big),$$

and the bending moment

$$m^+(w) = D\Big(æ\Delta w^+ + (1-æ)\,\frac{\partial^2 w^+}{\partial \nu^2}\Big),$$

where $\tau = (-\nu_2, \nu_1)$ and $\Delta w \equiv w_{,11} + w_{,22}$. Using the values of W^- and w^- and selecting the negative direction of the normal $(-\nu)$, we define $\sigma^-(W)$, $t^-(w)$, and $m^-(w)$ on γ^- similarly.

Let the solution of problem (2.1) be sufficiently smooth, so that

$$\sigma^\pm(W) \in L^2(\gamma) \times L^2(\gamma), \qquad t^\pm(w), m^\pm(w) \in L^2(\gamma). \tag{3.1}$$

As in [7], one can show that the conditions $[\sigma(W^*)] = [t(w^*)] = [m(w^*)] = 0$ are satisfied on γ. We denote $\sigma^* \equiv \sigma^\pm(W^*)$, $t^* \equiv t^\pm(w^*)$, and $m^* \equiv m^\pm(w^*)$.

For the case considered, the existence of the solution of problem (2.1) is equivalent [10] to the existence of the Lagrange factor $\xi \in L^2(\gamma)$ ($\xi \geqslant 0$ almost everywhere on γ) such that

$$J(W^*, w^*) - \Big\langle \xi, \Phi_z\Big(W^*, w^*, \frac{\partial w^*}{\partial \nu}\Big)\Big\rangle_\gamma \leqslant J(W, w) - \Big\langle \xi, \Phi_z\Big(W, w, \frac{\partial w}{\partial \nu}\Big)\Big\rangle_\gamma$$
$$\forall (W, w) \in H, \qquad \forall z,\ |z| \leqslant h. \tag{3.2}$$

Here

$$\Big\langle \xi, \Phi_z\Big(W^*, w^*, \frac{\partial \overset{*}{w}}{\partial \nu}\Big)\Big\rangle_\gamma = 0, \tag{3.3}$$

where $\langle \cdot, \cdot \rangle_\gamma$ denotes integration over the curve γ. Using the Green formula, one can write the minimum condition (3.2) in the equivalent form owing to the convexity of J and the linearity of Φ_z with respect to W and w:

$$\langle (AW^*, Bw^*) - (F, f), (W, w) \rangle_\Omega + \Big\langle (\sigma^*, t^*, m^*), \Big[W, w, \frac{\partial w}{\partial \nu}\Big]\Big\rangle_\gamma - \Big\langle \xi, \Phi_z\Big(W, w, \frac{\partial w}{\partial \nu}\Big)\Big\rangle_\gamma = 0$$
$$\forall (W, w) \in H, \qquad \forall z,\ |z| \leqslant h. \tag{3.4}$$

Here the operators A and B have the form $AW^* = -(\sigma_{1j,j}(W^*), \sigma_{2j,j}(W^*))$ and $Bw^* = D\Delta^2 w^*$. The variational equality (3.4) means that the equilibrium equations are satisfied everywhere in Ω_γ:

$$-\sigma_{ij,j}(W^*) = f_i, \qquad i = 1, 2, \qquad D\Delta^2 w^* = f, \tag{3.5}$$

and the following equality is satisfied almost everywhere in the curve γ:

$$(\sigma^*, t^*, m^*) \equiv \xi(n, z\cos\alpha). \tag{3.6}$$

Multiplying, in a scalar manner, both sides of (3.6) first by the vector $(n, z\cos\alpha)$ and then by an arbitrary orthogonal vector $(n, z\cos\alpha)_\perp$ and taking into account that $\xi \geqslant 0$, we obtain almost everywhere on γ $\forall z$ ($|z| \leqslant h$)

$$(\sigma^*, t^*, m^*) \cdot (n, z\cos\alpha) \geqslant 0, \qquad (\sigma^*, t^*, m^*) \cdot (n, z\cos\alpha)_\perp = 0. \tag{3.7}$$

With allowance for (3.6), equality (3.3) can be written as

$$\Big\langle (\sigma^*, t^*, m^*), \Big[W^*, w^*, \frac{\partial w^*}{\partial \nu}\Big]\Big\rangle_\gamma = 0. \tag{3.8}$$

Thus, if the solution $\{W^*, w^*\}$ of the equilibrium problem for a cracked plate is subject to condition (3.1), both formulations [in the form of the minimization problem (2.1) and the boundary-value problem (1.2), (3.5), (3.7), and (3.8)] are equivalent. Conditions (1.2) and (3.6) are linear with respect to the parameter z, $|z| \leqslant h$. This means that they should be satisfied for $z = h$ and $z = -h$ and, on the cut, they take the form

$$\sigma_{ij}(W^*)\nu_j\nu_i \tan\alpha + t^* = 0, \quad \sigma_{ij}(W^*)\nu_j\tau_i = 0,$$

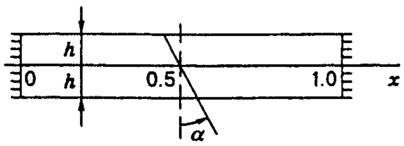

Fig. 2

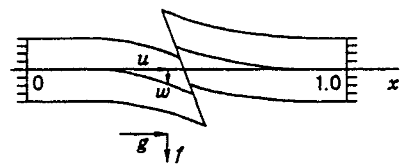

Fig. 3

$$\sigma_{ij}(W^*)\nu_j\nu_i([u_i^*]\nu_i + [w^*]\tan\alpha) + m^*\left[\frac{\partial w^*}{\partial \nu}\right] = 0, \tag{3.9}$$

$$-\sigma_{ij}(W^*)\nu_j\nu_i \geqslant \frac{1}{h}|m^*|, \qquad [u_i^*]\nu_i + [w^*]\tan\alpha \geqslant h\left|\left[\frac{\partial w^*}{\partial \nu}\right]\right|.$$

4. Formulation of the Beam Problem. We shall consider a homogeneous isotropic beam of unit length and thickness $2h$. Let the median line of the beam occupy the segment $(0,1)$ on the x axis. At the point $y = 0.5$, there is an oblique cut passing at the angle α to the vertical (Fig. 2). We assume that the cut does not reach the boundary, i.e., $0 \leqslant \tan\alpha < 1/2h$. We look for the functions $u(x)$ and $w(x)$ of the horizontal and vertical displacements of the median points under the external load [$g(x)$ and $f(x)$] (Fig. 3). The jam condition $u = w = w_x = 0$ at $x = 0$ and 1 is imposed at the external boundary. The condition of mutual nonpenetration of the cut faces takes the form $[u] + [w]\tan\alpha \geqslant h|[w_x]|$, where $[s]$ is the jump of the function s at the point y, i.e., $[s] = s(y+0) - s(y-0)$. The beam's energy functional is

$$J(u,w) = \int_0^1 \left(\frac{G}{2}u_x^2 + \frac{D}{2}w_{xx}^2 - gu - fw\right) dx.$$

We denote $\Omega = (0,y) \cup (y,1)$. We then introduce the Hilbert space

$$H = \{u \in H^1(\Omega), w \in H^2(\Omega) \,|\, u = w = w_x = 0 \text{ for } x = 0 \text{ and } 1\}$$

and the closed convex set

$$K = \{(u,w) \in H \,|\, [u] + [w]\tan\alpha \geqslant h|[w_x]|\}.$$

Let f and g be specified functions from $L_2(\Omega)$. The equilibrium problem of a beam with an oblique cut (2.1) is formulated as the variational inequality

$$\int_\Omega (Gu_x(\bar u_x - u_x) + Dw_{xx}(\bar w_{xx} - w_{xx}) - g(\bar u - u) - f(\bar w - w))\, dx \geqslant 0 \qquad \forall(\bar u, \bar w) \in K. \tag{4.1}$$

We introduce the differential formulation of problem (4.1). To do this, we use the following Green formula:

$$\int_\Omega (Gu_x\bar u_x + Dw_{xx}\bar w_{xx})dx = \int_\Omega (-Gu_{xx}\bar u + Dw_{xxxx}\bar w)dx - G[u_x\bar u] - D[w_{xx}\bar w_x] + D[w_{xxx}\bar w].$$

After that, the variational inequality (4.1) can be represented in the form

$$\int_\Omega ((-Gu_{xx} - g)(\bar u - u) + (Dw_{xxxx} - f)(\bar w - w))\, dx$$

$$- G[u_x(\bar u - u)] - D[w_{xx}(\bar w_x - w_x)] + D[w_{xxx}(\bar w - w)] \geqslant 0 \qquad \forall(\bar u, \bar w) \in K.$$

We take $\bar u = u + \xi$ and $\bar w = w + \eta$, where $\xi, \eta \in C_0^\infty(\Omega)$. Since $(u,w) \in K$ by definition and $(\xi,\eta) \in K$, choosing in turn $\xi = 0$ and $\eta = 0$, we obtain that the equations $-Gu_{xx} = g$ and $Dw_{xxxx} = f$ are fulfilled in the domain Ω. Now we take $\xi, \eta \in C_0^\infty(0,1)$ and $[\xi] = [\eta] = [\eta_x] = 0$ and $\xi(y) = a$, $\eta(y) = b$, and $\eta_x(y) = c$.

Choosing in turn the zero values for arbitrary a, b, and c, we have $[u_x] = [w_{xx}] = [w_{xxx}] = 0$. We define the auxiliary functions

$$\varphi^+(u,w) = [u] + [w]\tan\alpha + h[w_x], \qquad \varphi^-(u,w) = [u] + [w]\tan\alpha - h[w_x].$$

It follows that the initial nonpenetration condition is equivalent to the inequalities $\varphi^+(u,w) \geqslant 0$ and $\varphi^-(u,w) \geqslant 0$. Clearly, the constructed functions are linear in their arguments. We note that

$$[w_x] = \frac{\varphi^+(u,w) - \varphi^-(u,w)}{2h}, \qquad [u] + [w]\tan\alpha = \frac{\varphi^+(u,w) + \varphi^-(u,w)}{2}.$$

Taking into account the equations derived, we represent the variational inequality (4.1) in the new notation in the form

$$(Dw_{xxx}(y) + Gu_x(y)\tan\alpha)[\bar{w} - w] - \frac{1}{2}\Big(Gu_x(y) + \frac{1}{h}\,Dw_{xx}(y)\Big)\varphi^+(\bar{u} - u, \bar{w} - w)$$
$$-\frac{1}{2}\Big(Gu_x(y) - \frac{1}{h}\,Dw_{xx}(y)\Big)\varphi^-(\bar{u} - u, \bar{w} - w) \geqslant 0 \qquad \forall(\bar{u},\bar{w}) \in K. \tag{4.2}$$

Choosing $\bar{u} = u + \xi$ and $\bar{w} = w + \eta$ such that $\varphi^+(\xi,\eta) = 0$ and $\varphi^-(\xi,\eta) = 0$ for arbitrary $[\eta] = c$, we obtain $u_x(y)\tan\alpha + (D/G)w_{xxx}(y) = 0$. Now we choose ξ and η such that $\varphi^-(\xi,\eta) = 0$. For arbitrary $\varphi^+(\xi,\eta) = c \geqslant 0$, we have

$$-\frac{1}{2}\Big(Gu_x(y) + \frac{D}{h}\,w_{xx}(y)\Big)c \geqslant 0,$$

and, hence, $u_x(y) + (D/Gh)w_{xx}(y) \leqslant 0$ and $u_x(y) - (D/Gh)w_{xx}(y) \leqslant 0$. Since $(0,0) \in K$, one can use $(\bar{u},\bar{w}) = (0,0)$ in (4.2). This yields the inequality

$$\Big(Gu_x(y) + \frac{D}{h}\,w_{xx}(y)\Big)\varphi^+(u,w) + \Big(Gu_x(y) - \frac{D}{h}\,w_{xx}(y)\Big)\varphi^-(u,w) \geqslant 0.$$

Since $(u,w) \in K$, we have $2(u,w) \in K$. We take $(\bar{u},\bar{w}) = (2u,2w)$ and, substituting it into (4.2), we obtain

$$-\Big(Gu_x(y) + \frac{D}{h}\,w_{xx}(y)\Big)\varphi^+(u,w) - \Big(Gu_x(y) - \frac{D}{h}\,w_{xx}(y)\Big)\varphi^-(u,w) \geqslant 0.$$

The following relations follow from the last two inequalities and the fixed signs of the cofactors:

$$\Big(Gu_x(y) + \frac{D}{h}\,w_{xx}(y)\Big)\varphi^+(u,w) = 0, \qquad \Big(Gu_x(y) - \frac{D}{h}\,w_{xx}(y)\Big)\varphi^-(u,w) = 0.$$

Thus, we have proved the following theorem.

Theorem 1. *The variational inequality* (4.1) *is equivalent to the boundary-value problem*

$$-Gu_{xx} = g, \quad Dw_{xxxx} = f \quad \text{in} \quad \Omega, \qquad [u_x] = [w_{xx}] = [w_{xxx}] = 0,$$
$$u_x(y)\tan\alpha + \frac{h^2}{3}\,w_{xxx}(y) = 0, \qquad [u] + [w]\tan\alpha \geqslant h|[w_x]|, \qquad -u_x(y) \geqslant \frac{h}{3}\,|w_{xx}(y)|, \tag{4.3}$$
$$\Big(u_x(y) + \frac{h}{3}\,w_{xx}(y)\Big)([u] + [w]\tan\alpha + h[w_x]) = 0, \quad \Big(u_x(y) - \frac{h}{3}\,w_{xx}(y)\Big)([u] + [w]\tan\alpha - h[w_x]) = 0.$$

Adding the last two equalities, we obtain the equivalent relation

$$u_x(y)([u] + [w]\tan\alpha) + \frac{h^2}{3}\,w_{xx}(y)[w_x] = 0.$$

In this form, the boundary conditions of Theorem 1 are an analog of (3.9) for the one-dimensional domain.

5. Construction of the Solution of the Beam Problem. If we construct the solution of problem (4.3), we thus have the solution of the initial inequality (4.1) owing to its uniqueness. We note that it is convenient to seek the solution in the form of the sum $u = u^0 + u^1$ and $w = w^0 + w^1$ of the solutions of the inhomogeneous problem with zero boundary conditions

$$-Gu^0_{xx} = g, \qquad Dw^0_{xxxx} = f \qquad \text{in} \quad \Omega, \tag{5.1}$$

$$[u_x^0]=[w_{xx}^0]=[w_{xxx}^0]=0,\qquad u_x^0(y)=w_{xx}^0(y)=w_{xxx}^0(y)=0$$

and the homogeneous problem with nonzero conditions at the point y. It is clear that the solution $(u^0,w^0)\in (H^2(\Omega)\times H^4(\Omega))\cap H$ of problem (5.1) exists, and it is unique, since we solve as a matter of fact two independent problems on $(0,y)$ and $(y,1)$:

$$\begin{aligned} -Gu_{xx}^0=g,\quad Dw_{xxxx}^0=f \quad \text{in } (0,y),\qquad & -Gu_{xx}^0=g,\quad Dw_{xxxx}^0=f \quad \text{in } (y,1),\\ u^0(0)=w^0(0)=w_x^0(0)=0,\qquad & u^0(1)=w^0(1)=w_x^0(1)=0,\\ u_x^0(y)=w_{xx}^0(y)=w_{xxx}^0(y)=0,\qquad & u_x^0(y)=w_{xx}^0(y)=w_{xxx}^0(y)=0. \end{aligned}$$

We note that (u^0,w^0) is the solution of the problem of a beam with a cut for which the nonpenetration conditions are not imposed and the faces are assumed to be free.

For convenience, we introduce the following constants: $\delta=12h^2$ and $\rho=4h^2+\tan^2\alpha$.

Having solved problem (5.1), one can calculate the quantities

$$\varphi^+=[u^0]+[w^0]\tan\alpha+h[w_x^0],\qquad \varphi^-=[u^0]+[w^0]\tan\alpha-h[w_x^0],$$
$$\psi^+=[u^0]+[w^0]\tan\alpha+\frac{h\rho}{\delta}[w_x^0],\qquad \psi^-=[u^0]+[w^0]\tan\alpha-\frac{h\rho}{\delta}[w_x^0].$$

We introduce the following functions into our consideration:

$$\theta(x)=\begin{cases} x^2, & x\in(0,y),\\ (x-1)^2, & x\in(y,1),\end{cases}\qquad \theta_x(x)=\begin{cases} 2x, & x\in(0,y),\\ 2(x-1), & x\in(y,1),\end{cases}$$
$$\beta(x)=\begin{cases} 2x^3-3x^2, & x\in(0,y),\\ 2x^3-3x^2+1, & x\in(y,1).\end{cases}$$

The pairs (θ_x,θ) and (θ_x,β) belong to the space $(C^\infty(\Omega))^2\cap H$, and the relations

$$\theta_{xx}(x)\equiv 2,\quad \theta_{xxx}(x)\equiv 0,\quad [\theta]=0,\quad [\theta_x]=-2,\quad \beta_x(x)=6(x^2-x),$$
$$\beta_{xx}(x)=6(2x-1),\quad \beta_{xxx}(x)\equiv 12,\quad \beta_{xxxx}(x)\equiv 0,\quad [\beta]=1,\quad \beta_{xx}(y)=0,\quad [\beta_x]=0$$

hold. Here, as before, $y=0.5$.

Theorem 2. *The functions* $u(x)=u^0(x)+2h^2A\theta_x(x)$ *and* $w(x)=w^0(x)+6hB\theta(x)-A\beta(x)\tan\alpha$ *are the solution of the variational inequality* (4.1), *where*

$$(A,B)=\begin{cases} (0,0), & \text{if } \varphi^+\geqslant 0,\quad \varphi^-\geqslant 0,\\ (\delta+\rho)^{-1}(\varphi^+,\varphi^+), & \text{if } \varphi^+<0,\quad \psi^-\geqslant 0,\\ (\delta+\rho)^{-1}(\varphi^-,-\varphi^-), & \text{if } \varphi^-<0,\quad \psi^+\geqslant 0,\\ \big((\varphi^++\varphi^-)/2\rho,\ (\varphi^+-\varphi^-)/2\delta\big), & \text{if } \psi^+<0,\quad \psi^-<0.\end{cases}$$

Proof. It is sufficient to check conditions (4.3). Indeed, by virtue of the noted properties of the functions θ and β, we have

$$-Gu_{xx}=-Gu_{xx}^0-G2h^2A\theta_{xxx}=g-0=g \qquad \text{in } \Omega,$$
$$Dw_{xxxx}=Dw_{xxxx}^0+D6hB\theta_{xxxx}-DA\beta_{xxxx}\tan\alpha=f+0-0=f \qquad \text{in } \Omega,$$
$$[u_x]=[u_x^0]+2h^2A[\theta_{xx}]=0,\quad [w_{xx}]=[w_{xx}^0]+6hB[\theta_{xx}]-A[\beta_{xx}]\tan\alpha=0,$$
$$[w_{xxx}]=[w_{xxx}^0]+6hB[\theta_{xxx}]-A[\beta_{xxx}]\tan\alpha=0.$$

Now we calculate the following quantities for the functions constructed:

$$u_x(y)=u_x^0(y)+2h^2A\theta_{xx}(y)=4h^2A,\quad w_{xx}(y)=w_{xx}^0(y)+6hB\theta_{xx}(y)-A\beta_{xx}(y)\tan\alpha=12hB,$$
$$w_{xxx}(y)=w_{xxx}^0(y)+6hB\theta_{xxx}(y)-A\beta_{xxx}(y)\tan\alpha=-12A\tan\alpha.$$

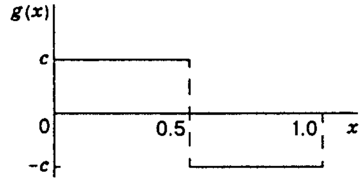

Fig. 4

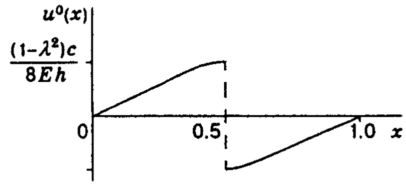

Fig. 5

Then

$$u_x(y) \pm \frac{h}{3} w_{xx}(y) = 4h^2(A \pm B), \qquad u_x(y)\tan\alpha + \frac{h^2}{3} w_{xxx}(y) = 0.$$

Next, $[u] = [u^0] - 4h^2 A$, $[w] = [w^0] - A\tan\alpha$, and $[w_x] = [w_x^0] - 12hB$, which yields $[u] + [w]\tan\alpha \pm h[w_x] = \varphi^\pm - \rho A \mp \delta B$.

Hence, it remains to check that

$$(A+B)(\varphi^+ - \rho A - \delta B) = 0, \qquad (A-B)(\varphi^- - \rho A + \delta B) = 0,$$
$$\varphi^+ \geqslant \rho A + \delta B, \qquad \varphi^- \geqslant \rho A - \delta B, \qquad -A \geqslant |B|.$$

Four variants are possible:

$$\begin{array}{lllll}
1) & A+B=0, & A-B=0, & \varphi^+ - \rho A - \delta B \geqslant 0, & \varphi^- - \rho A + \delta B \geqslant 0,\\
2) & A+B<0, & A-B=0, & \varphi^+ - \rho A - \delta B = 0, & \varphi^- - \rho A + \delta B \geqslant 0,\\
3) & A+B=0, & A-B<0, & \varphi^+ - \rho A - \delta B \geqslant 0, & \varphi^- - \rho A + \delta B = 0,\\
4) & A+B<0, & A-B<0, & \varphi^+ - \rho A - \delta B = 0, & \varphi^- - \rho A + \delta B = 0,
\end{array}$$

which give the desired values of the constants A and B. The theorem is proved.

Remark 1. It is easy to see that the solution (u, w) belongs to the space $(H^2(\Omega) \times H^4(\Omega)) \cap H$ owing to the smoothness of the functions u^0, w^0, θ, and β.

Remark 2. The constructed functions θ and β give a correction associated with the imposition of the nonpenetration condition for the solution (u^0, w^0) of the beam problem with free faces. Here $u = u^0$ and $w = w^0$ (i.e., $A = B = 0$) only in the case $\varphi^+ \geqslant 0$ and $\varphi^- \geqslant 0$.

Remark 3. Having found the solution of problem (4.1), we can calculate the remaining physical characteristics of the problem. Here the stresses and strains

$$\sigma(x) = æ\varepsilon(x) = æu_x(x) = æ(u_x^0(x) + 4h^2 A),$$

the bending moments

$$m(x) = Dæw_{xx}(x) = Dæ(w_{xx}^0(x) + 12hB - 6A(2x-1)\tan\alpha)$$

and the transverse forces

$$t(x) = Dw_{xxx}(x) = D(w_{xxx}^0(x) - 12A\tan\alpha)$$

are continuous functions on $(0, 1)$.

Now we deduce some corollaries from Theorem 2 for particular cases. Let $\alpha = 0$; then we have a vertical cut, and the nonpenetration condition takes the form $[u] \geqslant h|[w_x]|$. The corresponding boundary-value problem (4.3) for the variational inequality (4.1) takes the form

$$-Gu_{xx} = g, \qquad Dw_{xxxx} = f \qquad \text{in } \Omega,$$
$$[u_x] = [w_{xx}] = [w_{xxx}] = 0, \qquad w_{xxx}(y) = 0, \tag{5.2}$$

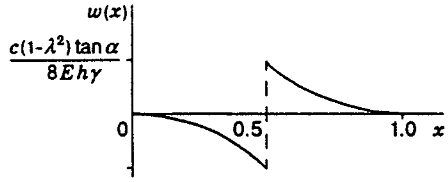

Fig. 6

$$\left(u_x(y) + \frac{h}{3}\,w_{xx}(y)\right)([u] + h[w_x]) = 0, \qquad \left(u_x(y) - \frac{h}{3}\,w_{xx}(y)\right)([u] - h[w_x]) = 0,$$

$$[u] \geqslant h|[w_x]|, \qquad -u_x(y) \geqslant \frac{h}{3}\,|w_{xx}(y)|.$$

According to Theorem 2, for the introduced quantities $\varphi^{\pm} = [u^0] \pm h[w_x^0]$ and $\varphi^{\pm} = [u^0] \pm (h/3)[w_x^0]$, the following corollary holds.

Corollary 1. *The functions* $u(x) = u^0(x) + (A/2)\theta_x(x)$ *and* $w(x) = w^0(x) + (3B/2h)\theta(x)$ *are the solution of problem* (5.2), *where*

$$(A,B) = \begin{cases} (0,0), & \text{if } \ \varphi^+ \geqslant 0, \ \ \varphi^- \geqslant 0, \\ (1/4)\,(\varphi^+, \varphi^+), & \text{if } \ \varphi^+ < 0, \ \ \psi^- \geqslant 0, \\ (1/4)\,(\varphi^-, -\varphi^-), & \text{if } \ \varphi^- < 0, \ \ \psi^+ \geqslant 0, \\ \big((\varphi^+ + \varphi^-)/2,\ (\varphi^+ - \varphi^-)/6\big), & \text{if } \ \psi^+ < 0, \ \ \psi^- < 0. \end{cases}$$

6. Beam with an Oblique Cut under Horizontal Loads. We assume that the vertical loads are zero, i.e., $f(x) \equiv 0$. We then obtain $w^0(x) \equiv 0$. Hence, $\varphi^+ = \varphi^- = \psi^+ = \psi^- = [u^0]$ and $B = 0$.

We denote the positive and negative parts of the number by plus and minus signs, respectively, i.e., $s = s^+ - s^-$, $s^+, s^- \geqslant 0$, and $s^+ s^- = 0$. From Theorem 2, one can deduce

Corollary 2. *For* $f = 0$, *the functions*

$$u(x) = u^0(x) - \frac{2h^2}{\rho}\,[u^0]^-\theta_x(x), \qquad w(x) = \frac{\tan\alpha}{\rho}\,[u^0]^-\beta(x)$$

are the solution of the variational inequality (4.1).

Thus, nonzero vertical displacements can arise in this case because of horizontal loads. For a vertical cut ($\alpha = 0$), the condition $f = 0$ entails $w = 0$.

As an illustration, we consider the external load specified as a function

$$g(x) = \begin{cases} c, & x \in (0; 0.5), \\ -c, & x \in (0.5; 1), \end{cases}$$

which corresponds to compression for $c \geqslant 0$ (Fig. 4). After that, one can calculate the function $u^0(x)$ (Fig. 5): $u^0(x) = [(1 - æ^2)/(2Eh)]x(1 - x)g(x)$. Its jump $[u^0] = -c(1 - æ^2)/(4Eh)$ is not positive, i.e., $[u^0]^- = c(1 - æ^2)/(4Eh)$. According to Corollary 2, we find a solution of the variational inequality (4.1) in the form

$$u(x) = \frac{c(1 - æ^2)}{2Eh}\begin{cases} -x^2 + (1 - \delta/6\rho)x, & x \in (0; 0.5), \\ x^2 - (1 + \delta/6\rho)x + \delta/6\rho, & x \in (0.5; 1), \end{cases}$$

$$w(x) = \frac{c(1 - æ^2)\tan\alpha}{4Eh\rho}\begin{cases} 2x^3 - 3x^2, & x \in (0; 0.5), \\ 2x^3 - 3x^2 + 1, & x \in (0.5; 1). \end{cases}$$

The graph of the function $w(x)$ is depicted in Fig. 6; we note that $[w] = c(1 - æ^2)\tan\alpha/(4Eh\rho)$. For extension (i.e., at $c < 0$), we obtain $[u^0] > 0$ and, hence, $[u^0]^- = 0$ and $u(x) = u^0(x)$ and $w(x) = 0$.

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